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# Bayesian Estimation of Three-Parameter Weibull Distribution under Asymmetric Loss Functions using Progressive Type-II Censored Data

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**Abstract**

We consider three-parameter Weibull distribution and derive Bayes estimators of the parameters under symmetric as well as asymmetric loss functions under progressive type-II censoring scheme. We use Lindley's approximation to obtain Bayes estimates and also perform simulation study for numerical illustrations.

**Keywords:** Bayes estimator; General Entropy loss function; LINEX loss function; Lindley's approximation; progressive type-II censoring; reliability function; squared-error loss function.

## 1. Introduction

The Weibull distribution is widely used as a lifetime model in reliability and survival analysis. Both, two and three parameter Weibull distributions have been extensively studied by several authors [see Cohen (1965, 1973, and 1975)]. Leone *et al.* (1960) obtained maximum likelihood estimators (MLEs), for the three parameter Weibull distribution using Newton Raphson Method. Harter and Moore (1965) provided iterative procedures for joint maximum likelihood estimation with complete and Type-II censored samples. Sinha and Solan(1988) considered this distribution and obtained Bayes estimates of parameters using Lindley's approximation. Kundu and Raqab (2009) considered the estimation of the stress strength model $P(X \leq Y)$ when X and Y are independent and both follow three parameter Weibull distributions.

A random variable X is said to follow the three parameter Weibull distribution if its probability density function (pdf) is given by

$$f(x;\beta,\theta,\gamma) = \frac{\beta}{\theta}(x-\gamma)^{\beta-1}\exp\left(-\frac{(x-\gamma)^{\beta}}{\theta}\right), \quad 0 \leq \gamma < x < \infty, \quad \beta,\theta > 0. \tag{1.1}$$

The reliability function and hazard rate functions for (1.1) at any time t, are given, respectively, by

$$R(t) = \exp\left( -\frac{(t-\gamma)^{\beta}}{\theta} \right); \qquad t > \gamma \tag{1.2}$$

and

$$h(t) = \frac{\beta}{\theta}(t-\gamma)^{\beta-1}; \qquad t > \gamma . \tag{1.3}$$

In lifetesting experiments, various kind of censoring schemes are frequently used to save time and cost. Among several existing censoring schemes, the progressive type-II censoring scheme has become very popular among workers in the field of reliability and survival analysis which is described as follows. Let $n$ items are put to test and the numbers $R_1$, $R_2, \dots, R_m$ are fixed such that, at the time of first failure, $R_1$ items are removed randomly from the experiment out of surviving n-1 items; at the time of second failure, $R_2$ items are removed out of $n$-2-$R_1$ surviving items; the process continue till the $m^{th}$ failure at which the test is terminated by removing the remaining $R_m (= n - m - \sum_{i=1}^{m} R_i)$ items. The observations $x_1$, $x_2, \dots, x_m$ are called progressively type-II censored order statistics with progressive censoring scheme $(R_1, R_2, \dots, R_m)$ . For the detailed description and related methodology see Balakrishnan and Aggarwala (2000).

In the present paper, we consider the problem of ML and Bayesian estimation of parameters and reliability function of the three-parameter Weibull distribution under progressive type-II censoring scheme. The rest of paper is organized as follows. In the section 2, we obtain ML estimates of parameters using Newton Raphson Method. In section 3, we consider the Bayesian estimation of parameters and reliability function under squared error loss function (SELF), general entropy loss function (GELF) and LINEX loss function using Lindley's approximation. In Section 4, we perform simulation study.

## 2. Maximum Likelihood estimation

Suppose $n$ identical items, the lifetime of each of which follow the pdf (1.1), are put to test. A sample of $m$ observations $x_1, x_2, \dots, x_m$ is obtained by following the progressive type-II censoring scheme $(R_1,R_2,\dots,R_m)$, described in Section 1. The likelihood function of the observed sample $d = (x_1, x_2, \dots, x_m)$ can be written as follows [see Balakrishnan and Aggarwala (2000).]

$$L(\beta,\theta,\gamma \mid d) = c\prod_{i=1}^{m} f(x_i;\beta,\theta,\gamma)\left(R(x_i;\beta,\theta,\gamma)\right)^{R_i}, \tag{2.1}$$

where, $c = n(n - R_1 - 1)(n - R_1 - R_2 - 2)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1)$.

Using (1.1) and (1.2), we obtain from (2.1), that

$$L(\beta,\theta,\gamma|d) = c\prod_{i=1}^{m}\frac{\beta}{\theta}(x_i-\gamma)^{\beta-1}\exp\left(-\left(\frac{(1+R_i)(x_i-\gamma)^{\beta}}{\theta}\right)\right).$$

$$= c\left(\frac{\beta}{\theta}\right)^{m}\left\{\prod_{i=1}^{m}(x_i-\gamma)^{\beta-1}\right\}\exp\left(-\frac{1}{\theta}\sum_{i=1}^{m}(1+R_i)(x_i-\gamma)^{\beta}\right). \tag{2.2}$$

Taking logarithm of both sides of (2.2), we obtain $\log L = l(say)$, given by

$$l \propto m\log(\beta) - m\log(\theta) + (\beta-1)\sum_{i=1}^{m}\log(x_i-\gamma) - \frac{1}{\theta}\sum_{i-1}^{m}(1+R_i)(x_i-\gamma)^{\beta} \tag{2.3}$$

In order to obtain MLEs of $\beta, \theta$ and $\gamma$ we differentiate (2.3) partially w.r.t parameters and get

$$\frac{\partial l}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m}\log(x_i-\gamma) - \frac{1}{\theta}\sum_{i=1}^{m}(R_i+1)(x_i-\gamma)^{\beta}\log(x_i-\gamma), \tag{2.4}$$

$$\frac{\partial l}{\partial \theta} = -\frac{m}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{m}(R_i+1)(x_i-\gamma)^{\beta} \tag{2.5}$$

and

$$\frac{\partial l}{\partial \gamma} = -(\beta-1)\sum_{i=1}^{m}(x_i-\gamma)^{-1} + \frac{\beta}{\theta}\sum_{i=1}^{m}(x_i-\gamma)^{\beta-1}. \tag{2.6}$$

We observe that the analytical solution of the likelihood equations (2.4), (2.5) and (2.6) is not possible. We, therefore, solve these equations using Newton-Raphson Method along with the condition that the MLE of $\gamma$ is less than or equal to $x_{(1)}$, the minimum of $(x_1, x_2, ..., x_m)$ and present the obtained estimates in various tables at the end of the paper.

## 3. Bayesian Estimation using Lindley's Approximation

We consider the following non-informative prior for shape parameters

$$\pi_1(\beta) \propto \frac{1}{\beta},$$

The gamma prior for the scale parameter given by

$$\pi_2(\theta) = \frac{\mu^{\nu}\theta^{\nu-1}}{\Gamma(\nu)}\exp(-\mu\theta)$$

and uniform prior for the location parameter

$$\pi_3(\gamma) \propto \frac{1}{c}\text{; where c is a constant.}$$

Assuming all the parameters independent, the joint prior distribution of $\beta, \theta$ and $\gamma$ can be written as follows

$$\pi(\beta,\theta,\gamma) = \pi_1(\beta)\pi_2(\theta)\pi_3(\gamma)$$

$$= \frac{\mu^{\nu}\,\theta^{\nu-1}}{\beta\, c\,\Gamma(\nu)}\exp(-\mu\theta);\ \beta,\theta,\upsilon,\mu > 0. \tag{3.1}$$

The posterior expectation of any parametric function of $\tau = (\beta,\theta,\gamma)$, say $\omega(\tau)$, can be obtained by solving the following ratio of integrals.

$$E(\omega(\tau)\,|\,d) = \frac{\int_{\tau}\omega(\tau)\pi(\tau)L(\tau\,|\,d)d\tau}{\int_{\tau}\pi(\tau)L(\tau\,|\,d)d\tau}, \tag{3.2}$$

which on using Lindley's approximation can be written in the following form.

$$\tilde{\omega}_S = \omega(\tau) + \psi + \omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3, \tag{3.3}$$

where,

$$\psi = (\delta_{12}\omega_{12} + \delta_{13}\omega_{13} + \delta_{23}\omega_{23}) + \frac{1}{2}(\delta_{11}\omega_{11} + \delta_{22}\omega_{22} + \delta_{33}\omega_{33}),$$

$$\eta_1 = (\delta_{11}g_1 + \delta_{12}g_2 + \delta_{13}g_3) + \frac{1}{2}(\delta_{11}\xi_1 + \delta_{12}\xi_2 + \delta_{13}\xi_3),$$

$$\eta_2 = (\delta_{21}g_1 + \delta_{22}g_2 + \delta_{23}g_3) + \frac{1}{2}(\delta_{21}\xi_1 + \delta_{22}\xi_2 + \delta_{23}\xi_3),$$

$$\eta_3 = (\delta_{31}g_1 + \delta_{32}g_2 + \delta_{33}g_3) + \frac{1}{2}(\delta_{31}\xi_1 + \delta_{32}\xi_2 + \delta_{33}\xi_3),$$

$$\xi_1 = \delta_{11}l_{111} + 2\delta_{12}l_{121} + 2\delta_{13}l_{131} + 2\delta_{23}l_{231} + \delta_{22}l_{221} + \delta_{33}l_{331},$$

$$\xi_2 = \delta_{11}l_{112} + 2\delta_{12}l_{122} + 2\delta_{13}l_{132} + 2\delta_{23}l_{232} + \delta_{22}l_{222} + \delta_{33}l_{332},$$

and

$$\xi_3 = \delta_{11}l_{113} + 2\delta_{12}l_{123} + 2\delta_{13}l_{133} + 2\delta_{23}l_{233} + \delta_{22}l_{223} + \delta_{33}l_{333},$$

All the subscripts 1, 2, 3 on the right hand sides refers to $\beta, \theta$ and $\gamma$ , respectively,

$$g_i = \frac{\partial g}{\partial \tau_i}, \quad \omega_i = \frac{\partial \omega(\tau)}{\partial \tau_i}, \; l_{ij} = \frac{\partial^2 l(\tau)}{\partial \tau_i \partial \tau_j}, \qquad i = 1,2,3$$

$$l_{abc} = \frac{\partial^3 l(\tau)}{\partial \beta^a \partial \theta^b \partial \gamma^c}, \; a,b,c = 0,1,2,3 \;\; and \;\; a+b+c = 3,$$

where $g = \log(\pi(\tau))$ and $\delta_{ij}$, is the *(i, j)*$^{th}$ element in the matrix $\{-l_{ij}\}^{-1}$, $i, j = 1,2,$ ; All the above expressions are to be evaluated at the MLE of the parameters.

### 3.1 Bayesian Estimation under SELF

Using the fact that, under SELF, the Bayes estimator of any parameter is its posterior mean, we obtain the Bayes estimators of $\beta, \theta, \gamma$ and R(t) by using (3.3) as follows.

1. When $\omega(\tau) = \beta$ , we have $\omega_1 = 1, \omega_2 = 0, \omega_3 = 0$ and $\psi = 0$ substituting these values in (3.4), we get the following Bayes estimates of $\beta$ .

$$\tilde{\beta}_s = \hat{\beta} + \eta_1. \tag{3.4}$$

2. When $\omega(\tau) = \theta$ , we have $\omega_1 = 0, \omega_2 = 1, \omega_3 = 0$ and $\psi = 0$. Thus from (3.5), we get the Bayes estimate of $\theta$ given by

$$\tilde{\theta}_s = \hat{\theta} + \eta_2 . \tag{3.5}$$

3. When $\omega(\tau) = \gamma$ , we have $\omega_1 = 0, \omega_2 = 0, \omega_3 = 1$ and $\psi = 0$. Then using (3.5), the Bayes estimates of $\gamma$ is given by

$$\tilde{\gamma}_s = \hat{\gamma} + \eta_3 . \tag{3.6}$$

4. When $\omega(\tau) = R(t) = \exp\left(\frac{-(t-\gamma)^{\beta}}{\theta}\right)$, we have

$$\tilde{R}_s(t) = \hat{R}(t) + \psi_R + \omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3 , \tag{3.7}$$

where,

$$\omega_1 = \frac{-(t-\gamma)^{\beta}}{\theta} \exp\left(\frac{-(t-\gamma)^{\beta}}{\theta}\right) \log(t-\gamma),$$

$$\omega_2 = \frac{(t-\gamma)^{\beta}}{\theta^2} \exp\left(\frac{-(t-\gamma)^{\beta}}{\theta}\right),$$

$$\omega_3 = \frac{\beta(t-\gamma)^{\beta-1}}{\theta} \exp\left(\frac{-(t-\gamma)^{\beta}}{\theta}\right)$$

and

$$\psi_R = \left(\frac{\delta_{12}\partial^2\omega}{\partial\beta\partial\theta} + \frac{\delta_{13}\partial^2\omega}{\partial\beta\partial\theta} + \frac{\delta_{23}\partial^2\omega}{\partial\theta\partial\gamma}\right) + \frac{1}{2}\left(\frac{\delta_{11}\partial^2\omega}{\partial\beta^2} + \frac{\delta_{22}\partial^2\omega}{\partial\theta^2} + \frac{\delta_{33}\partial^2\omega}{\partial\gamma^2}\right).$$

### 3.2 Bayesian Estimation under GELF

The General entropy loss function for estimating a parameter (parametric function) $\omega = \omega(\tau)$ by $\hat{\omega}$ proposed by calebria and pulcini (2006) is given by

$$L(\hat{\omega}, \omega) = \left(\frac{\hat{\omega}}{\omega}\right)^q - q\log\left(\frac{\hat{\omega}}{\omega}\right) - 1, \qquad q \neq 0 \tag{3.8}$$

The Bayes estimator of ω under GELF in (3.8) is

$$\widetilde{\omega}_G(\tau) = [E_\omega(\omega(\tau)^{-q})]^{-1/q}, \tag{3.9}$$

provided $E_\omega(\omega(\tau)^{-q})$ exist and finite. Thus, to obtain $\widetilde{\omega}_G(\tau)$,the Bayes estimate of $\omega(\pi)$ under GELF, we first find the posterior expectation $E_\omega(\omega(\tau)^{-q})$ for given $q$, using (3.3). Then the Bayes estimates of $\beta$ , $\theta$, $\gamma$ and and R(t) under GELF, can be obtained using (3.9).

1. When $\omega(\tau) = \beta^{-q}$, then using (3.5) we have

$$\widetilde{\beta}_G = (\hat{\beta}^{-q} + \psi_\beta + \omega_1\eta_1)^{-1/q}, \tag{3.10}$$

where, $\omega_1 = -q\beta^{-q-1}$ and $\psi_\beta = \dfrac{q(q-1)\delta_{11}\beta^{-q-2}}{2}$.

2. When $\omega(\tau) = \theta^{-q}$, then we have

$$\widetilde{\theta}_G = (\hat{\theta}^{-q} + \psi_\theta + \omega_2\eta_2)^{-1/q}, \tag{3.11}$$

where, $\omega_2 = -q\theta^{-q-1}$ and $\psi_\theta = \dfrac{q(q-1)\delta_{22}\theta^{-q-2}}{2}$.

3. When $\omega(\tau) = \gamma^{-q}$, then we have

$$\widetilde{\gamma}_G = (\hat{\gamma}^{-q} + \psi_\gamma + \omega_3\eta_3)^{-1/q} \tag{3.12}$$

where, $\omega_3 = -q\gamma^{-q-1}$ and $\psi_\gamma = \dfrac{q(q-1)\delta_{33}\gamma^{-q-2}}{2}$.

4. When $\omega(\tau) = R(t)^{-q} = \exp\left(\dfrac{q(t-\gamma)^\beta}{\theta}\right)$, we have

$$\widetilde{R}_G(t) = (\hat{R}(t)^{-q} + \psi'_R + \omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3)^{-1/q}, \tag{3.13}$$

where

$$\omega_1 = \frac{q(t-\gamma)^\delta}{\theta}\exp\left(\frac{q(t-\gamma)^\delta}{\theta}\right)\log(t-\gamma),$$

$$\omega_2 = \frac{-q(t-\gamma)^\delta}{\theta^2}\exp\left(\frac{q(t-\gamma)^\delta}{\theta}\right)\log(t-\gamma),$$

$$\omega_1 = \frac{q(t-\gamma)^\delta}{\theta}\exp\left(\frac{q(t-\gamma)^\delta}{\theta}\right)\log(t-\gamma),$$

and

$$\psi'_R = \left(\frac{\delta_{12}\partial^2\omega}{\partial\beta\partial\theta} + \frac{\delta_{13}\partial^2\omega}{\partial\beta\partial\theta} + \frac{\delta_{23}\partial^2\omega}{\partial\theta\partial\gamma}\right) + \frac{1}{2}\left(\frac{\delta_{11}\partial^2\omega}{\partial\beta^2} + \frac{\delta_{22}\partial^2\omega}{\partial\theta^2} + \frac{\delta_{33}\partial^2\omega}{\partial\gamma^2}\right).$$

### 3.3 Bayesian Estimation under LINEX Loss Function

The LINEX loss function, proposed by Zellner (1986), for estimating a parametric function $\omega(\tau)$ by its estimator $\tilde{\omega}(\tau)$ is given by

$$L(\hat{\omega}(\tau)-\omega(\tau)) = a\exp(c(\hat{\omega}(\tau)-\omega(\tau))) - c(\hat{\omega}(\tau)-\omega(\tau)) - 1, \qquad a>0,\ c\neq 0. \tag{3.14}$$

The Bayes Estimate of α under LINEX loss function is

$$\tilde{\omega}_l(\tau) = -\frac{1}{c}\log[E_{\omega(\tau)}(\exp(-c\omega(\tau)))], \tag{3.15}$$

Provided $E_{\omega(\tau)}(\exp(-c\omega(\tau)))$ exist and finite. Thus, to get $\tilde{\omega}(\tau)$, the Bayes estimate of $\omega(\tau)$ under LINEX loss function, we first obtain the posterior expectation $E_{\omega(\tau)}(\exp(-c\omega(\tau)))$ for given c, using (3.3), then the Bayes estimates of $\beta$, $\theta, \gamma$ and R(t) can be obtained using (3.15).

1. When $\omega(\tau) = \exp(-c\beta)$, then we have

$$\tilde{\beta}_l = -\frac{1}{c}\log E_\beta[\exp(-c\hat{\beta})],$$

$$= -\frac{1}{c}\log[\exp(-c\hat{\beta}) + \psi_\beta + \omega_1\eta_1], \tag{3.16}$$

where, $\omega_1 = -c\exp(-c\beta)$ and $\psi_\beta = \dfrac{c^2\delta_{11}\exp(-c\beta)}{2}$.

1. When $\omega(\tau) = \exp(-c\theta)$, then we have

$$\tilde{\theta}_l = -\frac{1}{c}\log E_0[\exp(-c\hat{\theta})],$$

$$= -\frac{1}{c}\log[\exp(-c\hat{\theta}) + \psi_\theta + \omega_2\eta_2], \tag{3.17}$$

where , $\omega_2 = -c\exp(-c\theta)$ and $\psi_\theta = \dfrac{c^2\delta_{22}\exp(-c\theta)}{2}$.

2. When $\omega(\tau) = \exp(-c\gamma)$, then we have

$$\gamma_l = -\frac{1}{c}\log E_\gamma[\exp(-c\hat{\gamma})],$$

$$= -\frac{1}{c}\log[\exp(-c\hat{\gamma}) + \psi_\gamma + \omega_3\eta_3], \tag{3.18}$$

where, $\omega_3 = -c\exp(-c\gamma)$ and $\psi_\gamma = \dfrac{c^2\delta_{33}\exp(-c\gamma)}{2}$.

3. When $\omega(\tau) = \exp(-cR(t)) = \exp\left(-c\exp\left(\dfrac{-(t-\gamma)^\beta}{\theta}\right)\right)$, then we have

$$\tilde{R}_l = -\frac{1}{c}\log E_R\left[\exp\left(-c\exp\left(\frac{-(t-\gamma)^\beta}{\theta}\right)\right)\right],$$

$$= -\frac{1}{c}\log\left(\exp\left(-c\exp\left(\frac{-(t-\gamma)^\beta}{\theta}\right)\right) + \psi''_R + \omega_1\eta_1 + \omega_2\eta_2 + \omega_2\eta_3\right), \tag{3.19}$$

$$\omega_1 = \frac{c(t-\gamma)^\beta}{\theta}\log(t-\lambda)\exp\left(-(1+c)\exp\left(\frac{-(t-\gamma)^\beta}{\theta}\right)\right),$$

$$\omega_2 = \frac{-c(t-\gamma)^{\beta}}{\theta^2}\exp\left(-(1+c)\exp\left(\frac{-(t-\gamma)^{\beta}}{\theta}\right)\right),$$

$$\omega_3 = \frac{-c\beta(t-\gamma)^{\beta}}{\theta}\exp\left(-(1+c)\exp\left(\frac{-(t-\gamma)^{\beta}}{\theta}\right)\right),$$

and

$$\psi''_R = \left(\frac{\delta_{12}\partial^2\omega}{\partial\beta\partial\theta} + \frac{\delta_{13}\partial^2\omega}{\partial\beta\partial\theta} + \frac{\delta_{23}\partial^2\omega}{\partial\theta\partial\gamma}\right) + \frac{1}{2}\left(\frac{\delta_{11}\partial^2\omega}{\partial\beta^2} + \frac{\delta_{22}\partial^2\omega}{\partial\theta^2} + \frac{\delta_{33}\partial^2\omega}{\partial\gamma^2}\right).$$

**3.4 Simulation Study**

In this section, we present simulation study for some numerical illustrations. For the values of the parameters $\beta$=2.5, $\theta$=2, and $\gamma$ =0.0001, we generate the type-II progressively censored samples using the algorithm of Balakrishnan and Sandu (2000) in software *R*. Since the evaluation of Bayes estimators through Lindley' approximation requires the MLEs of parameters, we first obtain the MLEs of $\beta$, $\theta$, and $\gamma$ using (2.4), (2.5) and (2.6), respectively. The obtained estimates are presented in the given tables. With the help of MLEs of parameters, we evaluate Bayes estimators taking the values of prior Hyper-parameters to be $\mu = 1$ and $\upsilon = 1$. In this study we generate 5000 samples and present the average estimates and corresponding RMSEs in various tables.

In this paper we have shown that the computations for the Bayes estimators for three-parameter family become easy with the present form of Lindley's approximation. Once the Bayesian estimation problem is set according to Section 3, we can compute the posterior expectation of any parametric function of parameters. The paper may be helpful for the workers in reliability and survival analysis who are not well aware with the advanced simulation techniques such as Markov Chain Monte Carlo, Gibbs Sampler etc.

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Table 1: Average Estimates and RMSEs (in the parenthesis) of β.

| Sample Size | | Progressive Censoring Scheme | $\hat{\beta}$ | $\tilde{\beta}_S$ | $\tilde{\beta}_G$ | | $\tilde{\beta}_l$ | |
|---|---|---|---|---|---|---|---|---|
| n | m | | | | q=-1 | q=1 | c=-1 | c=1 |
| 00 | 100 | All Items Failed | 1.8681 | 1.8785 | 1.8785 | 1.8786 | 1.8908 | 1.8660 |
| | | | (0.4327) | (0.4189) | (0.4189) | (0.4188) | (0.4046) | (0.4337) |
| | 80 | 20*1,0*40,0*39 | 1.8131 | 1.8271 | 1.8271 | 1.8272 | 1.8419 | 1.8120 |
| | | | (0.5059) | (0.4858) | (0.4858) | (0.4856) | (0.4673) | (0.5054) |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 0*39,0*40,20*1 | 1.5437 | 1.5636 | 1.5636 | 1.5641 | 1.5757 | 1.5514 |
| | | | (0.9518) | (0.9126) | (0.9126) | (0.9113) | (0.8910) | (0.9342) |
| | | 0*30,1*20, 0*30 | 1.8154 | 1.8282 | 1.8282 | 1.8283 | 1.8425 | 1.8135 |
| | | | (0.5031) | (0.4848) | (0.4848) | (0.4847) | (0.4668) | (0.5037) |
| | 60 | 40*1,0*30,0*29 | 1.7241 | 1.7452 | 1.7452 | 1.7455 | 1.7635 | 1.7260 |
| | | | (0.6347) | (0.6012) | (0.6012) | (0.6007) | (0.5752) | (0.6290) |
| | | 0*29,0*30,40*1 | 1.3018 | 1.2750 | 1.2750 | 1.3372 | 1.3501 | 1.3131 |
| | | | (1.5257) | (3.0299) | (3.0299) | (1.4532) | (1.4304) | (1.5391) |
| | | 0*10,1*40,0*10 | 1.7144 | 1.7342 | 1.7342 | 1.7345 | 1.7517 | 1.7161 |
| | | | (0.6530) | (0.6212) | (0.6212) | (0.6207) | (0.5962) | (0.6478) |
| | 40 | 60*1,0*20,0*19 | 1.5603 | 1.5883 | 1.5883 | 1.6007 | 1.6238 | 1.5733 |
| | | | (0.9121) | (1.7748) | (1.7748) | (0.8357) | (0.7961) | (0.8877) |
| | | 0*19,0*20,60*1 | 1.0808 | 1.2268 | 1.2268 | 1.2496 | 1.2346 | 1.2150 |
| | | | (2.0141) | (1.6211) | (1.6211) | (1.5636) | (1.6011) | (1.6511) |
| | | 0*10,3*20, 0*10 | 1.5799 | 1.6140 | 1.6140 | 1.6158 | 1.6374 | 1.5907 |
| | | | (0.8804) | (0.8225) | (0.8225) | (0.8136) | (0.7782) | (0.8576) |
| 60 | 60 | All Items Failed | 1.6828 | 1.7051 | 1.7051 | 1.7055 | 1.7226 | 1.6869 |
| | | | (0.7047) | (0.6671) | (0.6671) | (0.6665) | (0.6412) | (0.6949) |
| | 50 | 10*1,0*20,0*29 | 1.6187 | 1.6469 | 1.6469 | 1.6481 | 1.6673 | 1.6264 |
| | | | (0.8103) | (0.7633) | (0.7633) | (0.7578) | (0.7282) | (0.7942) |
| | | 1*10,0*20,0*20 | 1.6343 | 1.6608 | 1.6608 | 1.6613 | 1.6796 | 1.6412 |
| | | | (0.7849) | (0.7383) | (0.7383) | (0.7370) | (0.7083) | (0.7698) |
| | | 0*20,1*10, 0*20 | 1.6263 | 1.6502 | 1.6502 | 1.6538 | 1.6731 | 1.6323 |
| | | | (0.7996) | (0.8605) | (0.8605) | (0.7504) | (0.7198) | (0.7870) |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 40 | 20*1,0*20,0*19 | 1.5318 | 1.5522 | 1.5522 | 1.5739 | 1.5958 | 1.5469 |
| | | | (0.9684) | (2.3860) | (2.3860) | (0.8863) | (0.8476) | (0.9420) |
| | | 1*20,0*10,0*10 | 1.5736 | 1.6036 | 1.6036 | 1.6066 | 1.6272 | 1.5830 |
| | | | (0.8945) | (0.8644) | (0.8644) | (0.8324) | (0.7974) | (0.8749) |
| | | 2*10,0*20,0*10 | 1.5771 | 1.6085 | 1.6085 | 1.6100 | 1.6298 | 1.5874 |
| | | | (0.8871) | (0.8321) | (0.8321) | (0.8254) | (0.7920) | (0.8650) |

Note: $a$*$b$ indicates that $a$ is repeated $b$ times.

Table 2: Average Estimates and RMSEs (in the parenthesis) of $\theta$ ,

| Sample Size | | Progressive Censoring Scheme | $\hat{\theta}$ | $\tilde{\theta}_S$ | $\tilde{\theta}_G$ | | $\tilde{\theta}_l$ | |
|---|---|---|---|---|---|---|---|---|
| n | m | | | | q=-1 | q=1 | c=-1 | c=1 |
| 100 | 100 | All Items Failed | 1.1519 | 1.1671 | 1.1671 | 1.1673 | 1.1751 | 1.1588 |
| | | | (0.4599) | (0.4347) | (0.4347) | (0.4344) | (0.4229) | (.4471) |
| | 80 | 20*1,0*40,0*39 | 1.1273 | 1.1462 | 1.1462 | 1.1465 | 1.1556 | 1.1364 |
| | | | (0.8003) | (0.7682) | (0.7682) | (0.7676) | (0.7540) | (0.7832) |
| | | 0*39,0*40,20*1 | 1.2931 | 1.3108 | 1.3108 | 1.3110 | 1.3219 | 1.2992 |
| | | | (0.5279) | (0.5034) | (0.5034) | (0.5031) | (0.4894) | (0.5182) |
| | | 0*30,1*20, 0*30 | 1.1706 | 1.1872 | 1.1872 | 1.1874 | 1.1965 | 1.1776 |
| | | | (0.7294) | (0.7026) | (0.7026) | (0.7022) | (0.6892) | (0.7167) |
| | 60 | 40*1,0*30,0*29 | 1.0878 | 1.1127 | 1.1127 | 1.1133 | 1.1240 | 1.1009 |
| | | | (0.8681) | (0.8241) | (0.8241) | (0.8231) | (0.8063) | (0.8432) |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 0*29,0*30,40*1 | 1.5645 | 1.5730 | 1.5730 | 1.5781 | 1.5987 | 1.5549 |
| | | | (0.2594) | (0.2706) | (0.2706) | (0.2446) | (0.2349) | (0.2596) |
| | | 0*10,1*40,0*10 | 1.2135 | 1.2318 | 1.2318 | 1.2321 | 1.2444 | 1.2188 |
| | | | (0.6622) | (0.6340) | (0.6340) | (0.6335) | (0.6166) | (0.6523) |
| | 40 | 60*1,0*20,0*19 | 1.0204 | 1.0569 | 1.0569 | 1.0583 | 1.0708 | 1.0420 |
| | | | (0.9908) | (0.9224) | (0.9224) | (0.9198) | (0.8987) | (0.9484) |
| | | 0*19,0*20,60*1 | 2.0968 | 2.0675 | 2.0675 | 2.0679 | 2.1243 | 2.0138 |
| | | | (1.0094) | (1.0046) | (1.0046) | (1.0046) | (1.0154) | (1.0002) |
| | | 0*10,3*20, 0*10 | 1.3149 | 1.3331 | 1.3331 | 1.3331 | 1.3575 | 1.3070 |
| | | | (0.5281) | (0.5040) | (0.5040) | (0.5200) | (0.4742) | (0.5345) |
| 60 | 60 | All Items Failed | 1.0230 | 1.0468 | 1.0468 | 1.0474 | 1.0566 | 1.0365 |
| | | | (0.8931) | (0.8482) | (0.8482) | (0.8472) | (0.8316) | (0.8660) |
| | 50 | 10*1,0*20,0*29 | 0.9983 | 1.0267 | 1.0267 | 1.0276 | 1.0375 | 1.0152 |
| | | | (1.0383) | (0.9838) | (0.9838) | (0.9821) | (0.9647) | (1.0043) |
| | | 1*10,0*20,0*20 | 1.0103 | 1.0377 | 1.0377 | 1.0385 | 1.0486 | 1.0262 |
| | | | (1.0174) | (0.9652) | (0.9652) | (0.9638) | (0.9465) | (0.9854) |
| | | 0*20,1*10, 0*20 | 1.0463 | 1.0723 | 1.0723 | 1.0722 | 1.0833 | 1.0596 |
| | | | (0.9502) | (0.9072) | (0.9072) | (0.9029) | (0.8846) | (0.9241) |
| | 40 | 20*1,0*20,0*19 | 0.9683 | 1.0027 | 1.0027 | 1.0042 | 1.0156 | 0.9895 |
| | | | (1.0972) | (1.0340) | (1.0340) | (1.0268) | (1.0061) | (1.0541) |
| | | 1*20,0*10,0*10 | 1.0373 | 1.0675 | 1.0675 | 1.0679 | 1.0811 | 1.0527 |
| | | | (0.9709) | (0.9147) | (0.9147) | (0.9156) | (0.8922) | (0.9395) |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 2*10,0*20,0*10 | 1.0036 | 1.0356 | 1.0356 | 1.0364 | 1.0484 | 1.0218 |
| | | | (1.0341) | (0.9729) | (0.9729) | (0.9714) | (0.9509) | (0.9971) |

Table 3: Average Estimates and RMSEs (in the parenthesis) of $\gamma$

| Sample Size | | Progressive Censoring Scheme | $\hat{\gamma}$ | $\tilde{\gamma}_S$ | $\tilde{\gamma}_G$ | | $\tilde{\gamma}_l$ | |
|---|---|---|---|---|---|---|---|---|
| n | m | | | | q=-1 | q=1 | c=-1 | c=1 |
| 100 | 100 | All Items Failed | 0.1860 | 0.1860 | 0.1860 | 0.1860 | 0.1860 | 0.1860 |
| | | | (0.0405) | (0.0405) | (0.0405) | (0.0405) | (0.0405) | (0.0405) |
| | 80 | 20*1,0*40,0*39 | 0.1848 | 0.1848 | 0.1848 | 0.1848 | 0.1848 | 0.1848 |
| | | | (0.0408) | (0.0408) | (0.0408) | (0.0408) | (0.0408) | (0.0408) |
| | | 0*39,0*40,20*1 | 0.1858 | 0.1857 | 0.1857 | 0.1857 | 0.1857 | 0.1857 |
| | | | (0.0408) | (0.0408) | (0.0408) | (0.0408) | (0.0408) | (0.0408) |
| | | 0*30,1*20, 0*30 | 0.1860 | 0.1860 | 0.1860 | 0.1860 | 0.1860 | 0.1860 |
| | | | (0.0410) | (0.0410) | (0.0410) | (0.0410) | (0.0410) | (0.0410) |
| | 60 | 40*1,0*30,0*29 | 0.1852 | 0.1851 | 0.1851 | 0.1851 | 0.1851 | 0.1851 |
| | | | (0.0405) | (0.0405) | (0.0405) | (0.0405) | (0.0405) | (0.0405) |
| | | 0*29,0*30,40*1 | 0.1855 | 0.1856 | 0.1856 | 0.1856 | 0.1856 | 0.1856 |
| | | | (0.0409) | (0.0410) | (0.0410) | (0.0410) | (0.0410) | (0.0410) |
| | | 0*10,1*40,0*10 | 0.1859 | 0.1859 | 0.1859 | 0.1859 | 0.1859 | 0.1859 |
| | | | (0.0407) | (0.0407) | (0.0407) | (0.0407) | (0.0407) | (0.0407) |
| | 40 | 60*1,0*20,0*19 | 0.1851 | 0.1851 | 0.1851 | 0.1851 | 0.1851 | 0.1851 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | (0.0404) | (0.0404) | (0.0404) | (0.0404) | (0.0404) | (0.0404) |
| | | 0*19,0*20,60*1 | 0.1684 | 0.1681 | 0.1681 | 0.1681 | 0.1681 | 0.1681 |
| | | | (0.0583) | (0.0582) | (0.0582) | (0.0582) | (0.0582) | (0.0582) |
| | | 0*10,3*20, 0*10 | 0.1855 | 0.1855 | 0.1855 | 0.1855 | 0.1855 | 0.1855 |
| | | | (0.0407) | (0.0407) | (0.0407) | (0.0407) | (0.0407) | (0.0407) |
| 60 | 60 | All Items Failed | 0.2266 | 0.2266 | 0.2266 | 0.2266 | 0.2266 | 0.2266 |
| | | | (0.0610) | (0.0610) | (0.0610) | (0.0610) | (0.0610) | (0.0610) |
| | 50 | 10*1,0*20,0*29 | 0.2267 | 0.2266 | 0.2266 | 0.2266 | 0.2266 | 0.2266 |
| | | | (0.0608) | (0.0608) | (0.0608) | (0.0608) | (0.0608) | (0.0608) |
| | | 1*10,0*20,0*20 | 0.2282 | 0.2282 | 0.2282 | 0.2282 | 0.2282 | 0.2282 |
| | | | (0.0614) | (0.0614) | (0.0614) | (0.0614) | (0.0614) | (0.0614) |
| | | 0*20,1*10, 0*20 | 0.2277 | 0.2277 | 0.2277 | 0.2277 | 0.2277 | 0.2277 |
| | | | (0.0615) | (0.0615) | (0.0615) | (0.0615) | (0.0615) | (0.0615) |
| | 40 | 20*1,0*20,0*19 | 0.2267 | 0.2266 | 0.2266 | 0.2267 | 0.2266 | 0.2267 |
| | | | (0.0607) | (0.0607) | (0.0607) | (0.0608) | (0.0607) | (0.0607) |
| | | 1*20,0*10,0*10 | 0.2273 | 0.2273 | 0.2273 | 0.2273 | 0.2273 | 0.2273 |
| | | | (0.0611) | (0.0611) | (0.0611) | (0.0611) | (0.0611) | (0.0611) |
| | | 2*10,0*20,0*10 | 0.2297 | 0.2297 | 0.2297 | 0.2297 | 0.2297 | 0.2297 |
| | | | (0.0622) | (0.0622) | (0.0622) | (0.0622) | (0.0622) | (0.0622) |

Table 4: Average Estimates and RMSEs in the parenthesis of Reliability function at $t$=0.5.

| Sample Size n | m | Progressive Censoring Scheme | $\hat{R}(t)$ | $\widetilde{R}_S(t)$ | $\widetilde{R}_G(t)$ q=-1 | $\widetilde{R}_G(t)$ q=1 | $\widetilde{R}_l(t)$ c=-1 | $\widetilde{R}_l(t)$ c=1 |
|---|---|---|---|---|---|---|---|---|
| 100 | 100 | All Items Failed | 0.8427 | 0.8434 | 0.8427 | 0.8417 | 0.8440 | 0.8436 |
| | | | (0.0063) | (0.0062) | (0.0063) | (0.0065) | (0.0061) | (0.0062) |
| | 80 | 20*1,0*40,0*39 | 0.8313 | 0.8326 | 0.8317 | 0.8303 | 0.8334 | 0.8329 |
| | | | (0.0160) | (0.0157) | (0.0159) | (0.0162) | (0.0154) | (0.0156) |
| | | 0*39,0*40,20*1 | 0.8167 | 0.8189 | 0.8184 | 0.8172 | 0.8203 | 0.8198 |
| | | | (0.0229) | (0.0223) | (0.0224) | (0.0228) | (0.0219) | (0.0220) |
| | | 0*30,1*20, 0*30 | 0.8384 | 0.8393 | 0.8388 | 0.8377 | 0.8405 | 0.8400 |
| | | | (0.0076) | (0.0074) | (0.0075) | (0.0077) | (0.0072) | (0.0073) |
| | 60 | 40*1,0*30,0*29 | 0.8132 | 0.8157 | 0.8146 | 0.8125 | 0.8171 | 0.8163 |
| | | | (0.0201) | (0.0193) | (0.0196) | (0.0202) | (0.0189) | (0.0191) |
| | | 0*29,0*30,40*1 | 0.8084 | 0.7988 | 0.7986 | 0.8069 | 0.8097 | 0.8078 |
| | | | (0.0244) | (0.0835) | (0.0835) | (0.0267) | (0.0309) | (0.0318) |
| | | 0*10,1*40,0*10 | 0.8305 | 0.8318 | 0.8316 | 0.8304 | 0.8344 | 0.8338 |
| | | | (0.0126) | (0.0122) | (0.0123) | (0.0126) | (0.0117) | (0.0118) |
| | 40 | 60*1,0*20,0*19 | 0.7755 | 0.7786 | 0.7773 | 0.7768 | 0.7853 | 0.7835 |
| | | | (0.0413) | (0.1363) | (0.1368) | (0.0407) | (0.0374) | (0.0396) |
| | | 0*19,0*20,60*1 | 0.8250 | 0.8368 | 0.8370 | 0.8357 | 0.8413 | 0.8409 |
| | | | (0.0434) | (0.0408) | (0.0407) | (0.0410) | (0.0499) | (0.0400) |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 0*10, 3*20, 0*10 | 0.8244 | 0.8256 | 0.8263 | 0.8250 | 0.8321 | 0.8313 |
| | | | (0.0411) | (0.0407) | (0.0405) | (0.0409) | (0.0490) | (0.0491) |
| 60 | 60 | All Items Failed | 0.8264 | 0.8291 | 0.8281 | 0.8261 | 0.8305 | 0.8297 |
| | | | (0.0184) | (0.0176) | (0.0179) | (0.0185) | (0.0173) | (0.0175) |
| | 50 | 10*1,0*20,0*29 | 0.8125 | 0.8163 | 0.8151 | 0.8127 | 0.8182 | 0.8173 |
| | | | (0.0189) | (0.0185) | (0.0186) | (0.0189) | (0.0184) | (0.0181) |
| | | 1*10,0*20,0*20 | 0.8179 | 0.8215 | 0.8206 | 0.8183 | 0.8237 | 0.8227 |
| | | | (0.0290) | (0.0277) | (0.0280) | (0.0288) | (0.0270) | (0.0272) |
| | | 0*20,1*10, 0*20 | 0.8222 | 0.8244 | 0.8237 | 0.8225 | 0.8278 | 0.8267 |
| | | | (0.0261) | (0.0334) | (0.0336) | (0.0260) | (0.0242) | (0.0251) |
| | 40 | 20*1,0*20,0*19 | 0.7921 | 0.7926 | 0.7913 | 0.7938 | 0.8021 | 0.7997 |
| | | | (0.0433) | (0.1825) | (0.1830) | (0.0427) | (0.0390) | (0.0430) |
| | | 1*20,0*10,0*10 | 0.8116 | 0.8151 | 0.8147 | 0.8127 | 0.8197 | 0.8184 |
| | | | (0.0257) | (0.0262) | (0.0263) | (0.0254) | (0.0231) | (0.0238) |
| | | 2*10,0*20,0*10 | 0.8083 | 0.8128 | 0.8121 | 0.8096 | 0.8166 | 0.8154 |
| | | | (0.0205) | (0.0194) | (0.0196) | (0.0201) | (0.0182) | (0.0185) |